Virtual Population Analysis

Hiroshi Hakoyama*

January 5, 2015

1 Backward calculation

$$\begin{pmatrix}\text{Number alive}\\ \text{at beginning}\\ \text{of next year}\end{pmatrix} = \begin{pmatrix}\text{number alive}\\ \text{at beginning}\\ \text{of this year}\end{pmatrix} - \begin{pmatrix}\text{catch}\\ \text{this}\\ \text{year}\end{pmatrix} - \begin{pmatrix}\text{natural}\\ \text{mortality}\\ \text{this year}\end{pmatrix} \tag{1}$$

$$\begin{pmatrix}\text{Number alive}\\ \text{at beginning}\\ \text{of this year}\end{pmatrix} = \begin{pmatrix}\text{number alive}\\ \text{at beginning}\\ \text{of next year}\end{pmatrix} + \begin{pmatrix}\text{catch}\\ \text{this}\\ \text{year}\end{pmatrix} + \begin{pmatrix}\text{natural}\\ \text{mortality}\\ \text{this year}\end{pmatrix} \tag{2}$$

$$N_i = f(N_{i+1}, C_i, D_i) \tag{3}$$

For a single cohort of fish (all fish hatched at the same time), if we know the catch of each year $C_i (i = 1, 2, \ldots, t-1)$, the natural mortality D_i and the number of oldest fish N_t, we can calculate the number of fish each year N_i, starting from the oldest ages and moving backward to the youngest.

2 Discrete Fisheries

$$N_i = N_{i+1} + C_i + D_i \tag{4}$$

$$D_i = N_i(1 - s) \tag{5}$$

$$N_i = \frac{N_{i+1} + C_i}{s} \tag{6}$$

*FRA

3 Continuous-time fishing and natural mortality

3.1 Cohort dynamics

Let N be the population of a cohort of fish at time t. The rate of population decrease is expressed as follows:

$$\frac{dN}{dt} = -ZN, \tag{7}$$

where Z is the instantaneous coefficient of total mortality. The solution of Eq.(7) is

$$N_t = N_0 \exp\{-Zt\}, \tag{8}$$

or

$$N_t = N_i \exp\{-Z(t-i)\}. \tag{9}$$

Here we assume the mortality is decomposed into fishing mortality and natural mortality:

$$Z = F + M, \tag{10}$$

where F is the instantaneous coefficient of fishing mortality, and M is that of natural mortality. Both factors are independent.

3.2 Catch dynamics

Let h be the cumulative catch at time t. The dynamics of h is given by

$$\frac{dh}{dt} = FN. \tag{11}$$

The solution of Eq.(11) is given as follows:

$$\begin{aligned}
\int_{h_i}^{h_{i+1}} dh &= \int_i^{i+1} FN\,dt, \\
h_{i+1} - h_i &= F\int_i^{i+1} N_i \exp\{-Z(t-i)\}\,dt, \\
&= FN_i \frac{-1}{Z}[\exp\{-Z(t-i)\}]_i^{i+1}, \\
&= FN_i \frac{-1}{Z}(\exp\{-Z\} - 1),
\end{aligned}$$

$$C_i = h_{i+1} - h_i = \frac{F}{F+M} N_i (1 - \exp\{-Z\}), \tag{12}$$

where C_i is yearly catch; catch from time i to $i+1$.

3.3 Gulland's Virtual Population Analysis

The VPA model is based on following two formulae:

$$N_{i+1} = N_i \exp\{-(F_i + M)\}, \tag{13}$$

$$C_i = N_i \frac{F_i(1 - \exp\{-(F_i + M)\})}{F_i + M}. \tag{14}$$

In this case, we can not calculate the explicit function $N_i = f(N_{i+1}, C_i, M)$ from Eqs.(13, 14), so we have to find N_i numerically.

Note that we have a solution for F from Eq.(13),

$$F_i = -\log \frac{N_{i+1}}{N_i} - M. \tag{15}$$

4 Pope's approximation

Pope's approximation for the VPA model assumes instantaneous midyear fishery (Fig. 1).

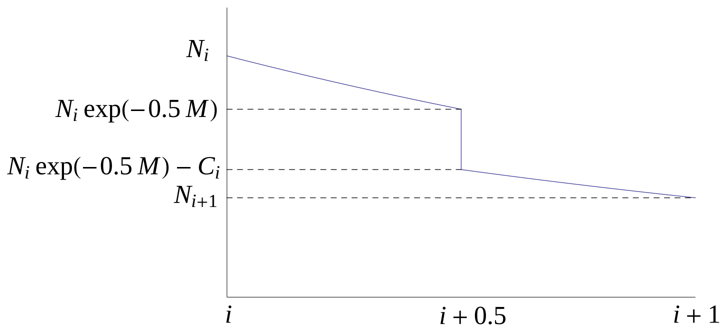

Figure 1: Instantaneous midyear fishery

Figure 1 indicates that the population size at time $i + 0.5$ multiplied by the survival rate equals the population size at time $i + 1$:

$$N_{i+1} = \left(N_i \exp\left\{-\frac{M}{2}\right\} - C_i\right) \exp\left\{-\frac{M}{2}\right\}. \tag{16}$$

Thus, $N_i = f(N_{i+1}, C_i, M)$ can be expressed as follows:

$$N_i = N_{i+1} \exp M + C_i \exp\left\{\frac{M}{2}\right\}. \tag{17}$$

5 Example: The Pacific stock of walleye pollock

5.1 Model

Here we consider multiple cohorts. Let $N_{a,i}$ be the number of fish at age a and year i.

$$N_{a+1,i+1} = N_{a,i} \exp\{-(F_{a,i} + M_a)\}, \tag{18}$$

$$N_{a,i} = N_{a+1,i+1} \exp M_a + C_{a,i} \exp\left\{\frac{M_a}{2}\right\}, \tag{19}$$

$$F_{a,l} = \frac{1}{3}(F_{a,l-3} + F_{a,l-2} + F_{a,l-1}), \tag{20}$$

$$F_{k,i} = F_{k-1,i}, \tag{21}$$

$$N_{a,l} = \frac{F_{a,l} + M_a}{F_{a,l}} C_{a,l} \frac{1}{(1 - \exp\{-(F_{a,l} + M_a)\})}, \tag{22}$$

$$N_{k,i} = \frac{F_{k+,i} + M_{k+}}{F_{k+,i}} C_{k+,i}, \tag{23}$$

where a is age ($a = 0, 1, 2, \ldots, k-1, k$). k is the last age of a year-class ($k = 8$). $k+$ indicates the plus group which includes older ages $k, k+1, k+2, \ldots$. i is year ($i = 1981, 1982, \ldots, 1998$). The latest year l is 1998. Appendix A shows the derivation of Eq.(23).

5.2 Procedure

- Assume arbitrary $F_{k+,l}$ (e.g., $F_{k+,l} = 1$)
- Calculate $N_{k,l} = \frac{F_{k+,l}+M_{k+}}{F_{k+,l}} C_{k+,l}$
- Calculate $N_{k-1,l-1}, N_{k-2,l-2}, \ldots$ using Eq.(19)
- Calculate $F_{k-1,l-1}, F_{k-2,l-2}, \ldots$ using $F_{a,i} = -\log \frac{N_{a+1,i+1}}{N_{a,i}} - M_a$
- Calculate $F_{k,l-1}$ using $F_{k,l-1} = F_{k-1,l-1}$
- Calculate $N_{k-1,l-2}, N_{k-2,l-3}, \ldots$ and $F_{k-1,l-2}, F_{k-2,l-3}, \ldots$
- Calculate all N and F to backward likewise
- Calculate $F_{k-1,l} = \frac{1}{3}(F_{k-1,l-1} + F_{k-1,l-2} + F_{k-1,l-3})$
- Calculate $N_{k-1,l} = \frac{F_{k-1,l}+M_{k-1}}{F_{k-1,l}} C_{k-1,l} \frac{1}{(1-\exp\{-(F_{k-1,l}+M_{k-1})\})}$
- Calculate all N and F to backward likewise
- Lastly calculate $F_{k,l}$ to be $F_{k,l} = F_{k-1,l}$ using excel solver

6 Appendix A

Let $N_{k+,t}$ be the number of fish of the plus group $k+$ at time t. $N_{k+,t}$ and $C_{k+,t}$ are expressed as:

$$N_{k+,t+1} = N_{k-1,t}\exp\{-(Z_{k-1})\} + N_{k,t}\exp\{-(Z_k)\} + N_{k+1,t}\exp\{-(Z_{k+1})\} + ..., \quad (24)$$

$$C_{k+,t+1} = C_{k,t} + C_{k+1,t} + C_{k+2,t} + ... \quad (25)$$

Using Eqs.(13,14), we have

$$N_i = \frac{C_i Z_i}{F_i} + N_{i+1}, \quad (26)$$

thus,

$$\begin{aligned} N_i &= \frac{C_i Z_i}{F_i} + N_{i+1} \\ &= \frac{C_i Z_i}{F_i} + \frac{C_{i+1} Z_{i+1}}{F_{i+1}} + N_{i+2} \\ &= \frac{C_i Z_i}{F_i} + \frac{C_{i+1} Z_{i+1}}{F_{i+1}} + \frac{C_{i+2} Z_{i+2}}{F_{i+2}} + N_{i+3} \end{aligned}$$

Note that $\lim_{k\to\infty} N_{i+k} = 0$. Therefore, if we assume that $F_i = F_{i+1} = ... = F_{i+}$, the number of fish of age i is expressed as:

$$\begin{aligned} N_i &= (C_i + C_{i+1} + ...)\frac{Z_{i+}}{F_{i+}} \\ &= C_{i+}\frac{Z_{i+}}{F_{i+}} \end{aligned}$$